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# Semilinear Duffing Equations Crossing Resonance Points*

Hao Dunyuan and Ma Shiwang

*Department of Mathematics, Inner Mongolia University, Hohhot, Inner Mongolia 010021, China*

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In this paper, using a generalized form of the Poincaré–Birkhoff theorem and a fixed point theorem, we prove, under weaker conditions, two theorems for the equation $\ddot{x} + g(x) = p(t)$, $p(t) \equiv p(t + 2\pi)$, of which one shows the existence of a



## 1. INTRODUCTION

We consider the Duffing equation

$$\ddot{x} + g(x) = p(t) \tag{1.1}$$

where $\ddot{x} = d^2x/dt^2$, $g, p \in C(R)$, $g(0) = 0$, $p(t)$ has the least positive period $2\pi$. Furthermore, we assume that the fundamental existence–uniqueness theorem holds for Eq. (1.1). The problem of the existence of a harmonic solution for Eq. (1.1) has been widely investigated in the literature and many results have been obtained. In studying Eq. (1.1), the auxiliary system

$$\dot{x} = y, \qquad \dot{y} = -g(x) \tag{1.2}$$

usually is needed. This is a planar autonomous system with orbit determined by the equations

$$V(x, y) = \tfrac{1}{2} y^2 + G(x) = c, \qquad G(x) = \int_0^x g(u)\, du, \qquad c > 0.$$

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We denote the curve $V^{-1}(c)$ by $\Gamma_c$ and the least positive period of the orbit $\Gamma_c$ by $\tau(c)$. Under the superlinear condition

$$\lim_{|x| \to \infty} x^{-1} g(x) = \infty, \tag{1.3}$$

Jacobowitz [13], Ding [10], Ding and Zanolin [6], Pei [18], and You [21] successively proved the existence of infinitely many harmonic solutions and subharmonic solutions for the Eq. (1.1).

Under the assumptions:

$H_1$. $g(x)$ is Lipschitz continuous; i.e., there exists a constant $K > 0$, such that

$$|g(x) - g(y)| \leqslant K\,|x - y| \qquad \text{for} \quad x, y \in R;$$

$H_2$. There exist two constants $A_0 > 0$, and $M_0 > 0$ such that

$$x^{-1} g(x) \geqslant A_0 \qquad \text{for} \quad |x| > M_0;$$

$H_3$. There exist a constant $\alpha > 0$, an integer $m > 0$, and two sequences $\{a_k\}$ and $\{b_k\}$ such that $a_k \to \infty$ and $b_k \to \infty$ as $k \to \infty$; moreover

$$\tau(a_k) < \frac{2\pi}{m} - \alpha, \qquad \tau(b_k) > \frac{2\pi}{m} + \alpha.$$

Ding [4] has proved that Eq. (1.1) has infinitely many harmonic solutions.

To obtain the above result, the author of [4] used the important technical condition $H_1$. In recent years, many authors have done a large quantity of work to remove this condition, see [5, 6, 8, 9, 19].

In this paper, in another direction and a way similar to that of [4], under the assumptions $H_1$ and $H_2$ and a weaker condition

$H'_3$. There exist an integer $m > 0$ such that

$$\limsup_{c \to \infty} \sqrt{c}\left(\tau(c) - \frac{2\pi}{m}\right) = +\infty, \qquad \liminf_{c \to \infty} \sqrt{c}\left(\tau(c) - \frac{2\pi}{m}\right) = -\infty, \tag{1.4}$$

we prove the following result:

THEOREM A. *If* $H_1$, $H_2$, *and* $H'_3$ *hold, then Eq.* (1.1) *has infinitely many harmonic solutions.*

The second result is on the existence of harmonic solutions of the Duffing equation crossing resonance points. Under the semilinear condition

$$0 \leqslant g_* = \liminf_{|x| \to \infty} x^{-1} g(x) \leqslant \limsup_{|x| \to \infty} x^{-1} g(x) = g^* < \infty \tag{1.5}$$

some investigations have been carried out by many authors; see [1–3, 5–9, 12, 14–17, 19]. All these studies focused on the time-map $\tau(c)$, and its asymptotic behavior at infinity is the crucial point for the existence of harmonic solution. The results obtained so far fail to answer the existence of harmonic solution with the case where $\lim_{c \to \infty} \tau(c) = 2\pi/m$, or $\tau(c)$ takes $2\pi/m$ as its limit point for some positive integer $m$. In this case, besides $H_1$ and $H_2$, we need the additional condition

$H_4$. For any positive integer $m$

$$\tau(c) \neq \frac{2\pi}{m} + O\left(\frac{1}{\sqrt{c}}\right) \qquad \text{as} \quad c \to \infty,$$

and we prove the following

THEOREM B. *If Eq.* (1.1) *satisfies* $H_1$, $H_2$, *and* $H_4$ *then* (1.1) *has at least one* $2\pi$*-period solution.*

The hypothesis $H_4$ is equivalent to $\lim_{c \to \infty} \sup |\tau(c) - (2\pi/m)| \sqrt{c} = \infty$ for any positive integer $m$. This means even if $\lim_{c \to \infty} \tau(c) = 2\pi/m$ for some $m$, we can still deduce the existence of a harmonic solution (1.1), whenever $\tau(c)$ converges to $2\pi/m$ slowly.

## 2. PRELIMINARIES

In proving Theorems A and B, we need the following lemmas.

LEMMA 2.1. *If* $H_2$ *holds, then there exist constants* $c_0 > 0$, *and* $A_1 > 0$ *such that whenever* $c \geqslant c_0$, $\Gamma_c$ *is a star-shaped curve about the origin* $O$ *and*

$$y^2 + xg(x) \geqslant A_1(x^2 + y^2) \qquad \textit{for} \quad (x, y) \in \Gamma_c. \tag{2.1}$$

*Proof.* This lemma has been proved in [4]. ∎

Now consider the equivalent system of Eq. (1.1):

$$\dot{x} = y, \qquad \dot{y} = -g(x) + p(t). \tag{2.2}$$

Let $\bar{x}(t)=\bar{x}(t, x_0, y_0)$, $\bar{y}(t)=\bar{y}(t, x_0, y_0)$ be the unique solution to (2.2) through the initial point $(\bar{x}(0)=x_0,\ \bar{y}(0)=y_0)$. Taking the transform $x(t)=r(t)\cos\theta$, $y(t)=r(t)\sin\theta$ we get the equations

$$\begin{aligned}\frac{dr}{dt}&=r\cos\theta\sin\theta-g(r\cos\theta)\sin\theta+p(t)\sin\theta\\ \frac{d\theta}{dt}&=-\sin^2\theta-\frac{1}{r}(g(r\cos\theta)\cos\theta-p(t)\cos\theta)\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\frac{dr}{dt}&=r\cos\theta\sin\theta-g(r\cos\theta)\sin\theta\\ \frac{d\theta}{dt}&=-\sin^2\theta-\frac{1}{r}g(r\cos\theta)\cos\theta\end{aligned}\tag{2.4}$$

from (2.2) and (1.2) respectively, for $r\neq 0$. Let

$$\bar{r}(t)=\bar{r}(t, r, \theta), \qquad \bar{\theta}(t)=\bar{\theta}(t, r, \theta)$$

be the solution of (2.3) through the initial point $\bar{r}(0)=r$, $\bar{\theta}(0)=\theta$, and

$$r_1(t)=r_1(t, r, \theta), \qquad \theta_1(t)=\theta_1(t, r, \theta)$$

be the solution of (2.4) through the same point $r_1(0)=r$, $\theta_1(0)=\theta$. We denote

$$\bar{\Theta}(t, \theta)=\bar{\theta}(2\pi, r, \theta)-\theta, \qquad \Theta_1(r, \theta)=\theta_1(2\pi, r, \theta)-\theta. \tag{2.5}$$

LEMMA 2.2. *If Eq.* (1.1) *satisfies* $H_1$ *and* $(r\cos\theta, r\sin\theta)\in\Gamma_c$, *then*

$$|\bar{\Theta}(r, \theta)-\Theta_1(r, \theta)|=O\left(\frac{1}{\sqrt{c}}\right) \qquad \textit{as} \quad c\to\infty. \tag{2.6}$$

*Proof.* Let $\bar{x}(t)$, $\bar{y}(t)$ and $x_1(t)$, $y_1(t)$ be solutions of (2.2) and (1.2), respectively, through the same initial point $(x, y)\in\Gamma_c$ at $t=0$; then

$$\begin{aligned}\bar{x}(t)-x_1(t)&=\int_0^t(\bar{y}(s)-y_1(s))\,ds\\ \bar{y}(t)-y_1(t)&=\int_0^t[-g(\bar{x}(s))+g(x_1(s))]\,ds+\int_0^t p(s)\,ds.\end{aligned}$$

Using $H_1$ we have

$$
\begin{aligned}
|\bar{x}(t) - x_1(t)| &\leqslant \int_0^t |\bar{y}(s) - y_1(s)|\, ds \\
|\bar{y}(t) - y_1(t)| &\leqslant K \int_0^t |\bar{x}(s) - x_1(s)|\, ds + Mt,
\end{aligned}
\tag{2.7}
$$

where $M = \max_{t \in [0, 2\pi]} p(t)$. Let $L = \max(K, 1)$. It follows from (2.7) that

$$
\begin{aligned}
&|\bar{x}(t) - x_1(t)| + |\bar{y}(t) - y_1(t)| \\
&\qquad \leqslant L \int_0^t (|\bar{x}(s) - x_1(s)| + |\bar{y}(s) - y_1(s)|)\, ds + Mt.
\end{aligned}
$$

By the Gronwall inequality [20],

$$
|\bar{x}(t) - x_1(t)| + |\bar{y}(t) - y_1(t)| \leqslant M \int_0^t \exp(L(t-s))\, ds \leqslant \frac{M}{L} e^{Lt}.
$$

Moreover,

$$
[(\bar{x}(t) - x_1(t))^2 + (\bar{y}(t) - y_1(t))^2]^{1/2} \leqslant \frac{M}{L} e^{Lt}.
$$

In particular

$$
[(\bar{x}(2\pi) - x_1(2\pi))^2 + (\bar{y}(2\pi) - y_1(2\pi))^2]^{1/2} \leqslant \frac{M}{L} e^{2\pi L}. \tag{2.8}
$$

On the other hand, by Lemma 2.3 below,

$$
r_1(2\pi) = [(x_1(2\pi))^2 + (y_1(2\pi))^2]^{1/2} \geqslant \frac{\sqrt{c}}{\sqrt{L}}. \tag{2.9}
$$

The inequalities (2.8) and (2.9) show that the origin must be outside the disc

$$
[(x - x_1(2\pi))^2 + (y - y_1(2\pi))^2]^{1/2} \leqslant \frac{M}{L} e^{2\pi L}
$$

and

$$
|\bar{\theta}(2\pi, r, \theta) - \theta_1(2\pi, r, \theta)| < \frac{\pi}{2}
$$

provided $c$ is large enough. Therefore

$$|\bar{\Theta}(r, \theta) - \Theta_1(r, \theta)| = |\bar{\theta}(2\pi, r, \theta) - \theta_1(2\pi, r, \theta)| \leqslant \arcsin \left[ \frac{M}{L} e^{2\pi L} / r_1(2\pi) \right]$$

$$\leqslant \arcsin \frac{M e^{2\pi L}}{\sqrt{L}\sqrt{c}} = O\left(\frac{1}{\sqrt{c}}\right) \qquad \text{as} \quad c \to \infty.$$

This completes the proof of Lemma 2.2. ∎

LEMMA 2.3. *If* $H_1$ *holds,* $L = \max(K, 1)$, *and* $(x, y) = (r\cos\theta, r\sin\theta) \in \Gamma_c$ *then* $r \geqslant \sqrt{c}/\sqrt{L}$.

*Proof.* From $\frac{1}{2}y^2 + G(x) = c$, and $|g(x)| \leqslant K|x|$, it follows that

$$c = \tfrac{1}{2}y^2 + \int_0^x g(s)\, ds \leqslant \tfrac{1}{2}y^2 + \tfrac{1}{2}Kx^2 \leqslant L(x^2 + y^2) = Lr^2.$$

Therefore $r \geqslant \sqrt{c/L}$. ∎

COROLLARY. *Let* $T\colon R^2 \to R^2$ *be defined by* $\bar{x} = \bar{x}(2\pi, x, y)$, $\bar{y} = \bar{y}(2\pi, x, y)$, *and let* $D_c$ *be the set bounded by the curve* $\Gamma_c$, *then the origin* $O \in T(D_c)$ *for* $c$ *large enough.*

*Proof.* Consider the map $T_1\colon R^2 \to R^2$ defined by

$$x_1 = x_1(2\pi, x, y), \qquad y_1 = y_1(2\pi, x, y).$$

It is clear that $T_1(\Gamma_c) = \Gamma_c$. By Lemmas 2.2 and 2.3, $T(\Gamma_c)$ is a curve surrounding the origin provided $c$ is large enough. $T$ is an area-preserving homeomorphism. Thus $O \in T(D_c)$. ∎

LEMMA 2.4. *If* $H_1$, $H_2$ *hold, then*

$$\liminf_{c \to \infty} \tau(c) \geqslant \frac{4}{\sqrt{A_0}} \arcsin \sqrt{\frac{A_0}{K}} > 0,$$

*where* $K$, *being the same as in* $H_1$, *can be chosen large enough to make* $A_0/K \leqslant 1$.

*Proof.* Let $G(h(c)) = G(-h_1(c)) = c$, $h(c), h_1(c) > 0$; then $\tau(c) = \int_{-h_1(c)}^{h(c)} \sqrt{2}\,[c - G(u)]^{-1/2}\, du$.

Assume $c$ is large enough so that $h(c), h_1(c) > M_0$. For $z > M_0$

$$\begin{aligned}
G(z) &= \int_0^z g(u)\,du = \int_0^{M_0} g(u)\,du + \int_{M_0}^z g(u)\,du \\
&\geqslant \int_0^{M_0} g(u)\,du + \int_{M_0}^z A_0 u\,du = \int_0^{M_0} g(u)\,du + \tfrac{1}{2}A_0(z^2 - M_0^2) \\
&= B + \tfrac{1}{2}A_0 z^2,
\end{aligned}$$

where $B = \int_0^{M_0} g(u)\,du - \frac{1}{2}A_0 M_0^2$. Since $G(h(c)) = c = \int_0^{h(c)} g(u)\,du \leqslant \int_0^{h(c)} Ku\,du = \frac{1}{2}K[h(c)]^2$, $h(c) \geqslant \sqrt{2c/K}$. Moreover,

$$\begin{aligned}
\int_{M_0}^{h(c)} \frac{du}{\sqrt{c - G(u)}} &\geqslant \int_{M_0}^{h(c)} \left[ c - B - \frac{1}{2}A_0 u^2 \right]^{-1/2} du \\
&= \sqrt{\frac{2}{A_0}} \operatorname{arc\,sin}\left( \sqrt{\frac{A_0}{2(c-B)}} \cdot u \right)\Bigg|_{M_0}^{h(c)} \\
&\geqslant \sqrt{\frac{2}{A_0}} \left[ \operatorname{arc\,sin} \sqrt{\frac{A_0 c}{K(c-B)}} - \operatorname{arc\,sin} \sqrt{\frac{A_0}{2(c-B)}}\, M_0 \right]. \qquad (2.11)
\end{aligned}$$

Similarly, $G(z) \geqslant -D + \frac{1}{2}A_0 z^2$, where $D = \frac{1}{2}A_0 M_0^2 + \int_{-M_0}^0 g(u)\,du$, $h_1(c) \geqslant \sqrt{2c/K}$, and

$$\int_{-h_1(c)}^{-M_0} \frac{du}{\sqrt{c - G(u)}} \geqslant \sqrt{\frac{2}{A_0}} \left[ \operatorname{arc\,sin} \sqrt{\frac{A_0 c}{K(c+D)}} - \arcsin \sqrt{\frac{A_0}{2(c+D)}}\, M_0 \right]. \qquad (2.12)$$

Combining inequalities (2.11) and (2.12), we have

$$\begin{aligned}
\tau(c) &= \sqrt{2} \int_{-h_1(c)}^{h(c)} \frac{du}{\sqrt{c - G(u)}} \\
&= \sqrt{2} \left[ \left( \int_{-h_1(c)}^{-M_0} + \int_{-M_0}^{M_0} + \int_{M_0}^{h(c)} \right) \frac{du}{\sqrt{c - G(u)}} \right] \\
&\geqslant \frac{2}{\sqrt{A_0}} \left[ \operatorname{arc\,sin} \sqrt{\frac{A_0 c}{K(c+D)}} - \operatorname{arc\,sin} \sqrt{\frac{A_0}{2(c+D)}}\, M_0 \right. \\
&\qquad \left. + \operatorname{arc\,sin} \sqrt{\frac{A_0 c}{K(c-B)}} - \operatorname{arc\,sin} \sqrt{\frac{A_0}{2(c-B)}}\, M_0 \right] \\
&\qquad + \sqrt{2} \int_{-M_0}^{M_0} \frac{du}{\sqrt{c - G(u)}}.
\end{aligned}$$

As $c \to \infty$, the terms on the right side of the above inequality go to zero except for

$$\arc\sin \sqrt{\frac{A_0 c}{K(c+D)}} \qquad \text{and} \qquad \arc\sin \sqrt{\frac{A_0 c}{K(c-B)}}.$$

So we have

$$\lim_{c\to\infty} \inf \tau(c) \geqslant \frac{4}{\sqrt{A_0}} \arc\sin \sqrt{\frac{A_0}{K}} > 0. \quad \blacksquare$$

The following lemma is devoted to proving Theorem A.

LEMMA 2.5. *If* $H_1$, $H_2$, $H_3'$ *hold then there exist two sequence* $\{a_k\}$, $\{b_k\}$ *such that*

$$\Theta_1(r,\theta) \leqslant -2m\pi - mA_1 k/\sqrt{a_k} \qquad \textit{for} \quad (r\cos\theta, r\sin\theta) \in \Gamma_{a_k},$$
$$\Theta_1(r,\theta) \geqslant -2m\pi + mA_1 k/\sqrt{b_k} \qquad \textit{for} \quad (r\cos\theta, r\sin\theta) \in \Gamma_{b_k},$$

*where* $m>0$ *and* $A_1$ *are the same as in* $H_3'$ *and Lemma* 2.1 *respectively.*

*Proof.* Let $(r\cos\theta, r\sin\theta) \in \Gamma_{a_k}$. Consider the solution $r_1(t,r,\theta)$, $\theta_1(t,r,\theta)$ of Eq. (2.4). It follows from Lemma 2.1 and the second equation of (2.4) that

$$\frac{d}{dt}\theta_1(t,r,\theta) \leqslant -A_1 \tag{2.13}$$

provided $a_k \geqslant c_0$. Denote $\Theta_1(r,\theta) = \theta_1(2\pi, r, \theta) - \theta = -2l\pi - \sigma$, where $l \geqslant 0$ is an integer and $0 \leqslant \sigma < 2\pi$. Let $t_\sigma$ be the time duration in which $\theta_1(t)$ decreases from $\theta - 2l\pi$ to $\theta - 2l\pi - \sigma$. Thus $l \cdot \tau(a_k) + t_\sigma = 2\pi$. Since $0 \leqslant t_\sigma < \tau(a_k)$ we have

$$2\pi = l\cdot\tau(a_k) + t_\sigma < (l+1)\cdot\tau(a_k).$$

By the second equality of (1.4) in $H_3'$, there exists a sequence $\{a_k\}$ such that $a_k \to \infty$, $k/\sqrt{a_k} \to 0$, as $k \to \infty$ and

$$\sqrt{a_k}\left(\tau(a_k) - \frac{2\pi}{m}\right) \leqslant -k.$$

Hence

$$\tau(a_k) \leqslant \frac{2\pi}{m} - \frac{k}{\sqrt{a_k}}.$$

This means that $l \geqslant m$. If $l \geqslant m+1$ then

$$\Theta_1(r, \theta) \leqslant -2l\pi \leqslant -2(m+1)\pi \leqslant -2m\pi - mA_1 k/\sqrt{a_k} \tag{2.14}$$

whenever $k$ is large enough. If $l=m$ then

$$t_\sigma = 2\pi - m\tau(a_k) \geqslant 2\pi - m\left(\frac{2\pi}{m} - \frac{k}{\sqrt{a_k}}\right) = \frac{mk}{\sqrt{a_k}}.$$

Using (2.13) we have

$$-\sigma = \int_{l\tau(a_k)}^{l\tau(a_k)+t_\sigma} \frac{d}{dt}\theta_1(t, r, \theta)\, dt \leqslant -A_1 t_\sigma \leqslant -\frac{mA_1 k}{\sqrt{a_k}}.$$

Furthermore

$$\Theta_1(r, \theta) = -2l\pi - \sigma \leqslant -2m\pi - mA_1 k/\sqrt{a_k}. \tag{2.15}$$

Combining (2.14), (2.15) yields the first inequality of this lemma. The second inequality can be proved in a similar way. ∎

The following two lemmas are devoted to proving Theorem B.

LEMMA 2.6. *If* $\mathrm{H}_1$, $\mathrm{H}_2$, $\mathrm{H}_4$ *hold then there exists a sequence* $\{c_k\}$, $c_k \to \infty$ *as* $k \to \infty$ *such that* $\sqrt{c_k} \geqslant k^2$ *and*

$$\tau(c_k) \geqslant 2\pi + \frac{k}{\sqrt{c_k}} \tag{2.16}$$

*or*

$$\frac{2\pi}{m+1} + \frac{k}{\sqrt{c_k}} \leqslant \tau(c_k) \leqslant \frac{2\pi}{m} - \frac{k}{\sqrt{c_k}}, \tag{2.17}$$

*where* $m$ *is some positive integer.*

*Proof.* Since $\tau(c) \neq 2\pi/n + O(1/\sqrt{c})$ for any positive integer $n$ and

$$\liminf_{c \to \infty} \tau(c) \geqslant \frac{4}{\sqrt{A_0}} \arcsin \sqrt{\frac{A_0}{K}} > 0,$$

the asymptotic behaviour of $\tau(c)$ is one of the following three cases.

*Case* 1.

$$\lim_{c\to\infty} \tau(c) = \frac{2\pi}{m} \qquad \text{for some positive integer } m. \tag{2.18}$$

This means $\tau(c)$ tends to $2\pi/m$ but slower than $2\pi/m + O(1/\sqrt{c})$ does. From $H_4$, for any positive integer $k$ there is $c_k$ such that $\sqrt{c_k}\,|\tau(c_k) - (2\pi/m)| \geqslant k^2 > k$ and $\sqrt{c_k} \geqslant k^2$. Thus, there exists a subsequence of $\{c_k\}$, still denoted as $\{c_k\}$, such that

$$\tau(c_k) \geqslant \frac{2\pi}{m} + \frac{k}{\sqrt{c_k}} \tag{2.19}$$

or

$$\tau(c_k) \leqslant \frac{2\pi}{m} - \frac{k}{\sqrt{c_k}}. \tag{2.20}$$

If $m=1$ and (2.19) holds, then the conclusion of Lemma 2.6 is proved. If $m=1$ and (2.20) holds, noting $k^2 \leqslant \sqrt{c_k}$ and $\lim_{c\to\infty} \tau(c) = 2\pi$, we have $2\pi/2 + k/\sqrt{c_k} \leqslant \tau(c_k) \leqslant 2\pi - k/\sqrt{c_k}$. This is (2.17). If $m \geqslant 2$, from (2.18) and (2.19)

$$\frac{2\pi}{m} + \frac{k}{\sqrt{c_k}} \leqslant \tau(c_k) \leqslant \frac{2\pi}{m-1} - \frac{k}{\sqrt{c_k}}$$

for $k$ sufficiently large. This shows (2.17) holds for $m-1$. From (2.18) and (2.20) it follows that

$$\frac{2\pi}{m+1} + \frac{k}{\sqrt{c_k}} \leqslant \tau(c_k) \leqslant \frac{2\pi}{m} - \frac{k}{\sqrt{c_k}}$$

for $k$ sufficiently large. Thus, Lemma 2.6 is proved for Case 1.

*Case* 2.

$$\lim_{c\to\infty} \tau(c) = \lambda > 0, \qquad \lambda \neq \frac{2\pi}{n} \qquad \text{for any positive integer } n.$$

In this case, we can choose $c_k$ such that

$$\lambda - \frac{1}{k} \leqslant \tau(c_k) \leqslant \lambda + \frac{1}{k} \qquad \text{and} \qquad \sqrt{c_k} > k^2$$

for $k$ large enough. If $\lambda > 2\pi$, then $\tau(c_k) \geqslant \lambda - 1/k \geqslant 2\pi + (\lambda - 2\pi - 1/k) \geqslant 2\pi + k/\sqrt{c_k}$ for $k$ large enough and (2.16) holds. If $2\pi/(m+1) < \lambda < 2\pi/(m)$ for some $m$, then we have, for $k$ large enough,

$$\frac{2\pi}{m+1} + \frac{k}{\sqrt{c_k}} \leqslant \frac{2\pi}{m+1} + \left(\lambda - \frac{2\pi}{m+1} - \frac{1}{k}\right)$$
$$\leqslant \tau(c_k) \leqslant \frac{2\pi}{m} + \left(\lambda - \frac{2\pi}{m} + \frac{1}{k}\right) \leqslant \frac{2\pi}{m} - \frac{k}{\sqrt{c_k}}.$$

This is (2.17) and Lemma 2.6 is proved for Case 2.

*Case* 3.

$$\lim_{c \to \infty} \tau(c) \text{ does not exist.}$$

If $\lim_{c \to \infty} \tau(c) = \infty$, then we can choose $c_k$ such that $\sqrt{c_k} \geqslant k^2$ and $\tau(c_k) \geqslant 2\pi + k/\sqrt{c_k}$. If $\lim_{c \to \infty} \inf \tau(c) = \lambda < \infty$, then by Lemma 2.4 $\lambda > 0$. We can choose a sequence $c_k \to \infty$ as $k \to \infty$ such that $\lim_{k \to \infty} \tau(c_k) = \bar{\lambda} > \lambda$, $\bar{\lambda} \neq 2\pi/n$ for any positive integer $n$, and $\sqrt{c_k} \geqslant k^2$. If $\bar{\lambda} > 2\pi$, then (2.16) holds. If $2\pi/(m+1) < \bar{\lambda} < 2\pi/m$ for some positive integer $m$, in the way similar to Case 2, we have (2.17) hold. Now the proof of Lemma 2.6 is completed. ∎

LEMMA 2.7. *If* $H_1$, $H_2$, $H_4$ *hold then there exists a sequence* $\{c_k\}$, $c_k \to \infty$ *as* $k \to \infty$ *such that* $\sqrt{c_k} \geqslant k^2$ *and*

$$-A_1 2\pi \geqslant \Theta_1(r, \theta) \geqslant -2\pi + \frac{A_1 k}{\sqrt{c_k}} \tag{2.21}$$

*or*

$$-2m\pi - \frac{mA_1 k}{\sqrt{c_k}} \geqslant \Theta_1(r, \theta) \geqslant -2\pi(m+1) + \frac{(m+1)\, A_1 k}{\sqrt{c_k}}, \tag{2.22}$$

*where* $m$ *is some positive integer,* $A_1 \geqslant 0$ *is given in Lemma* 2.1, *and* $(r\cos\theta, r\sin\theta) \in \Gamma_{c_k}$.

*Proof.* Let $\{c_k\}$ be the sequence in Lemma 2.6 and $c_k \geqslant c_0$, where $c_0$ is given in Lemma 2.1. For $(r\cos\theta, r\sin\theta) \in \Gamma_{c_k}$, $(d/dt)\, \theta_1(t, r, \theta) \leqslant -A_1$. Let $\Theta_1(r, \theta) = -2l\pi - \sigma$, where $l \geqslant 0$ is an integer, $0 \leqslant \sigma < 2\pi$, and $t_\sigma$ denote the time duration in which $\theta_1(t, r, \theta)$ decreases from $\theta - 2l\pi$ to $\theta - 2l\pi - \sigma$, then $l\tau(c_k) + t_\sigma = 2\pi$, and $0 \leqslant t_\sigma < \tau(c_k)$.

If (2.17) holds then $l = m$ follows from

$$l\left(\frac{2\pi}{m+1}+\frac{k}{\sqrt{c_k}}\right) \leqslant l\tau(c_k) \leqslant l\tau(c_k)+t_\sigma$$
$$= 2\pi \leqslant (l+1)\,\tau(c_k) \leqslant (l+1)\left(\frac{2\pi}{m}-\frac{k}{\sqrt{c_k}}\right).$$

Since

$$\tau(c_k)-t_\sigma = (l+1)\,\tau(c_k)-[l\tau(c_k)+t_\sigma] = (l+1)\,\tau(c_k)-2\pi$$
$$\geqslant (l+1)\left(\frac{2\pi}{m+1}+\frac{k}{\sqrt{c_k}}\right)-2\pi,$$

we have $\tau(c_k)-t_\sigma \geqslant (m+1)\,k/\sqrt{c_k}$, and

$$-2\pi+\sigma = \int_{l\tau(c_k)+t_\sigma}^{(l+1)\,\tau(c_k)} \frac{d}{dt}\,\theta_1(t, r, \theta)\,dt \leqslant -A_1(\tau(c_k)-t_\sigma)$$
$$\leqslant -A_1(m+1)\,k/\sqrt{c_k}.$$

Thus

$$\Theta_1(r, \theta) = -2l\pi-\sigma = -2(l+1)\,\pi+2\pi-\sigma$$
$$\geqslant -2(m+1)\,\pi+\frac{A_1(m+1)\,k}{\sqrt{c_k}}. \tag{2.23}$$

On the other hand, $t_\sigma = 2\pi - l\tau(c_k) \geqslant 2\pi - m(2\pi/m - k/\sqrt{c_k}) = mk/\sqrt{c_k}$, so that

$$-\sigma = \int_{l\tau(c_k)}^{l\tau(c_k)+t_\sigma} \frac{d}{dt}\,\theta_1(t, r, \theta)\,dt \leqslant -A_1 t_\sigma \leqslant -\frac{A_1 mk}{\sqrt{c_k}}.$$

Thus

$$\Theta_1(r, \theta) = -2l\pi-\sigma \leqslant -2m\pi-\frac{A_1 mk}{\sqrt{c_k}}. \tag{2.24}$$

The inequalities (2.23) and (2.24) show that (2.22) is true.

If (2.16) holds, then $l = 0$ follows from $\tau(c_k) \geqslant 2\pi + k/\sqrt{c_k}$, $\tau(c_k) > t_\sigma \geqslant 0$ and $2\pi = l\tau(c_k)+t_\sigma \geqslant l(2\pi + k/\sqrt{c_k})$. According to its definition

$$\Theta_1(r, \theta) = \int_0^{2\pi} \frac{d}{dt}\,\theta_1(t, r, \theta)\,dt \leqslant -A_1 2\pi,$$

$$\begin{aligned}\Theta_1(r, \theta) &= -2\pi - \int_{2\pi}^{\tau(c_k)} \frac{d}{dt}\,\theta_1(t, r, \theta)\,dt \\ &\geqslant -2\pi - (-A_1)(\tau(c_k) - 2\pi) \geqslant -2\pi + A_1 k/\sqrt{c_k}.\end{aligned}$$

These two inequalities show that (2.21) is true, and the proof of Lemma 2.7 is completed. ∎

## 3. THE PROOFS OF THEOREMS A AND B

Define the mapping $T: R^2 \to R^2$ as

$$T(x, y) = (\bar{x}(2\pi, x, y), \bar{y}(2\pi, x, y)), \tag{3.1}$$

where $\bar{x}(t, x, y)$ and $\bar{y}(t, x, y)$ is the solution of Eq. (2.2) starting at $(x, y)$. It is well known that $T$ is an area-preserving diffeomorphism and the fixed point of $T$ corresponds to a $2\pi$-period solution of (1.1). Mapping $T$ has the polar coordinate expression, through Eqs. (2.3) and (2.5)

$$T(r, \theta) = (r^* = \bar{r}(2\pi, r, \theta), \quad \theta^* = \bar{\theta}(2\pi, r, \theta) = \theta + \bar{\Theta}(r, \theta)). \tag{3.2}$$

It is easy to see that if $\bar{r}(t, r, \theta) > 0$ for $t \in [0, 2\pi]$ then $\bar{\theta}(2\pi, r, \theta)$ is continuous in $(r, \theta)$; furthermore

$$\bar{\theta}(2\pi, r, \theta + 2\pi) = \bar{\theta}(2\pi, r, \theta) + 2\pi. \tag{3.3}$$

In order to prove Theorem A, we need a generalized version of the Poincaré–Birkhoff twist theorem due to Ding [10, 11] and cited in [7]. A simple expression of the generalization of the twist theorem goes as follows.

Twist Theorem. *Let $T: R^2 \to R^2$ be a area-preserving homeomorphism and let $D_1$, $D_2$ be two open domains bounded by $\Gamma_1$ and $\Gamma_2$ star-shaped curves about the origin $O$, respectively, such that $O \in D_1 \subset \bar{D}_1 \subset D_2$. If the polar coordinate expression of $T$,*

$$\bar{r} = f(r, \theta), \qquad \bar{\theta} = \theta + g(r, \theta),$$

*satisfies the twist condition $g(r, \theta) > 0$ on $\Gamma_1$ and $g(r, \theta) < 0$ on $\Gamma_2$, then $T$ has at least two fixed points in $B = D_2 \backslash \bar{D}_1$.*

*The Proof of Theorem A.* Assume $H_1$, $H_2$, $H'_3$ hold. By Lemma 2.5, we can select a sequence $\{a_k\}$, $a_k \to \infty$ as $k \to \infty$ such that

$$\Theta_1(r, \theta) \leqslant -2m\pi - mA_1 k/\sqrt{a_k} \tag{3.4}$$

for $(r\cos\theta, r\sin\theta) \in \Gamma_{a_k}$ and $k$ sufficiently large. On the other hand, it follows from Lemma 2.2 that

$$|\bar{\Theta}(r, \theta) - \Theta_1(r, \theta)| = O\left(\frac{1}{\sqrt{a_k}}\right) \qquad \text{as} \quad k \to \infty \tag{3.5}$$

for $(r\cos\theta, r\sin\theta) \in \Gamma_{a_k}$. Hence, from (3.4), (3.5) we have

$$\bar{\Theta}(r, \theta) = \Theta_1(r, \theta) + O\left(\frac{1}{\sqrt{a_k}}\right) \leqslant -2m\pi - mA_1 \frac{k}{\sqrt{a_k}} + O\left(\frac{1}{\sqrt{a_k}}\right),$$

so that

$$\bar{\Theta}(r, \theta) \leqslant -2m\pi - \frac{1}{\sqrt{a_k}}(mA_1 k + O(1)). \tag{3.6}$$

As $k$ going large

$$\frac{1}{\sqrt{a_k}}(mA_1 k + O(1)) > 0. \tag{3.7}$$

Thus

$$\bar{\Theta}(r, \theta) < -2m\pi \tag{3.8}$$

holds for $(r\cos\theta, r\sin\theta) \in \Gamma_{a_k}$ and $k$ large enough. Also using Lemma 2.5 and carrying out the argument similar to above we can obtain a sequence $\{b_k\}$ such that $b_k \to \infty$ as $k \to \infty$, and

$$\bar{\Theta}(r, \theta) > -2m\pi \tag{3.9}$$

for $(r\cos\theta, r\sin\theta) \in \Gamma_{b_k}$ and $k$ large enough. There is no loss in generality in assuming that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k < \cdots.$$

Let $D_{1k}$ and $D_{2k}$ denote the open sets bounded by the star-shaped closed curves $\Gamma_{a_k}$ and $\Gamma_{b_k}$ respectively. It is clear that $O \in D_{1k} \subset \bar{D}_{1k} \subset D_{2k}$. Let $A_k = \bar{D}_{2k} - D_{1k}$. Thus, when $k$ is sufficiently large, we have

$$T_k\colon A_k \to T(A_k) \subset R^2 - \{O\}, \qquad O \in T(D_{1k}), \tag{3.10}$$

where $T_k$ is the restriction of $T$ on $A_k$. The inequalities (3.8), (3.9), and (3.10) show that the conditions of the twist theorem are satisfied. It follows from the twist theorem that there exist at least two fixed points of $T_k$ on $A_k$. Therefore, there is a point $(x^*, y^*) \in A_k$ such that

$$x^* = \bar{x}(2\pi, x^*, y^*), \qquad y^* = \bar{y}(2\pi, x^*, y^*),$$

which is a $2\pi$-period solution of Eq. (1.1). Thus (1.1) has infinitely many harmonic solutions. ∎

In order prove Theorem B we use the following fixed point theorem due to Ding [3].

FIXED POINT THEOREM. *Let $B \subset R^2$ be a compact domain with star-shaped boundary about the origin $O$, and $T$: $B \to R^2$ be a continuous mapping. If for any $p \in \partial B$ and $\lambda \geqslant 1$, $\overline{OT(p)} \neq \lambda\, \overline{Op}$, then there exists at least one fixed point $p_0 \in B$ for $T$.*

*The Proof of Theorem B.* If $H_1$, $H_2$, $H_4$ hold, then it follows from Lemma 2.7 that there exists a sequence $\{c_k\}$, with $c_k \to \infty$ as $k \to \infty$, such that (2.21) or (2.22) is true for $(r\cos\theta, r\sin\theta) \in \Gamma_{c_k}$. Since $|\bar{\Theta}(r, \theta) - \Theta_1(r, \theta)| = O(1/\sqrt{c})$ as $c \to \infty$ (see Lemma 2.2), there exist $E > 0$, $k_0 > 0$ such that

$$\Theta_1(r, \theta) - \frac{E}{\sqrt{c_k}} < \bar{\Theta}(r, \theta) < \Theta_1(r, \theta) + \frac{E}{\sqrt{c_k}} \tag{3.11}$$

for $(r\cos\theta, r\sin\theta) \in \Gamma_{c_k}$ and $k \geqslant k_0$. By (2.21), (2.22), and (3.11) we have

$$\begin{aligned} -2A_1\pi + \frac{E}{\sqrt{c_k}} &\geqslant \Theta_1(r, \theta) + \frac{E}{\sqrt{c_k}} > \bar{\Theta}(r, \theta) > \Theta_1(r, \theta) - \frac{E}{\sqrt{c_k}} \\ &\geqslant -2\pi + \frac{A_1 k}{\sqrt{c_k}} - \frac{E}{\sqrt{c_k}}, \end{aligned}$$

or

$$\begin{aligned} &-2(m+1)\,\pi + \frac{(m+1)\,A_1 k}{\sqrt{c_k}} - \frac{E}{\sqrt{c_k}} \\ &\quad \leqslant \Theta_1(r, \theta) - \frac{E}{\sqrt{c_k}} \leqslant \bar{\Theta}(r, \theta) \\ &\quad < \Theta_1(r, \theta) + \frac{E}{\sqrt{c_k}} \leqslant -2m\pi - \frac{mkA_1}{\sqrt{c_k}} + \frac{E}{\sqrt{c_k}}. \end{aligned}$$

If we choose $k_1$ so large that $mA_1k_1 > E$, $E/\sqrt{c_{k_1}} < 2A_1\pi$, $A_1k_1 > E$, and recall $\sqrt{c_k} \geqslant k^2$ in Lemma 2.7, then it is clear that

$$0 > \bar{\Theta}(r, \theta) > -2\pi$$

or

$$-2(m+1)\,\pi < \bar{\Theta}(r, \theta) < -2m\pi$$

for $(r\cos\theta, r\sin\theta) \in \Gamma_{c_{k_1}}$.

Let $A$ denote the region surrounded by $\Gamma_{c_{k_1}}$. Consider the Poincaré mapping defined by Eq. (1.1) and restricted to $\bar{A}$. For any $p \in \Gamma_{c_{k_1}}$, $\overline{OT(p)} \neq \lambda\,\overline{Op}$ for any $\lambda > 0$. By the fixed point theorem, $T|_{\bar{A}}$ has at least one fixed point $(x^*, y^*) \in A$. The solution $x(t, x_0 = x^*, \dot{x}_0 = y^*)$ is a $2\pi$-period solution of Eq. (1.1).

## 4. DISCUSSION

Consider an example

$$\ddot{x} + (m+1)^2\,x - \arctan x = 4\cos(m+1)\,t, \tag{4.1}$$

where $m$ is a positive integer. This equation, studied in [3], does not have a $2\pi$-period solution. Let $g(x) = (m+1)^2\,x - \arctan x$ and $\tau(c)$ stands for the period of orbit $\frac{1}{2}y^2 + \int_0^x g(u)\,du = c$, which corresponds to the equation $\ddot{x} + g(x) = 0$. It can be shown that

$$\tau(c) - \frac{2\pi}{m+1} = O\left(\frac{1}{\sqrt{c}}\right). \tag{4.2}$$

This means that Theorem B could not be sharpened in this paper's manner. We prove (4.2) as follows. From

$$g'(x) = (m+1)^2 - \frac{1}{1+x^2},$$

$$\frac{1}{2}\,m^2x^2 \leqslant \int_0^x g(u)\,du \leqslant \frac{1}{2}\,(m+1)^2\,x^2,$$

and

$$\tfrac{1}{2}y^2 + \tfrac{1}{2}m^2x^2 \leqslant \tfrac{1}{2}y^2 + \int_0^x g(u)\,du = c < \tfrac{1}{2}y^2 + \tfrac{1}{2}(m+1)^2\,x^2$$

it follows that

$$\tfrac{1}{2}r^2 \leqslant c \leqslant \tfrac{1}{2}(m+1)^2\, r^2$$

where $r^2 = x^2 + y^2$. Furthermore, for any $(x, y) \in \Gamma_c$,

$$\sqrt{2c} \geqslant r \geqslant \frac{\sqrt{2c}}{m+1}. \tag{4.3}$$

Under the polar coordinate expression of equation $\ddot{x} + g(x) = 0$,

$$\tau(c) = \int_{-2\pi}^{0} \left[ \sin^2\theta + (m+1)^2 \cos^2\theta - \frac{1}{r}\, h(r\cos\theta)\cos\theta \right]^{-1} d\theta$$

where $h(x) = \arctan x$, $(r, \theta) \in \Gamma_c$. Using $\int_{-2\pi}^{0} [\sin^2\theta + (m+1)^2 \cos^2\theta]^{-1}\, d\theta = 2\pi/(m+1)$, we have

$$\tau(c) - \frac{2\pi}{m+1} = \int_{-2\pi}^{0} \frac{1}{rH(\theta)}\, h(r\cos\theta)\cos\theta\, d\theta, \tag{4.4}$$

where

$$H(\theta) = \left[ \sin^2\theta + (m+1)^2\cos^2\theta - \frac{1}{r}\, h(r\cos\theta)\cos\theta \right] \times [\sin^2\theta + (m+1)^2\cos^2\theta].$$

Since $(1/r)\, h(r\cos\theta)\cos\theta \to 0$ as $c \to \infty$ and $(1/r)\, h(r\cos\theta)\cos\theta \geqslant 0$,

$$\begin{aligned} \tau(c) - \frac{2\pi}{m+1} &\geqslant \int_{-2\pi}^{0} \frac{(1/r)\, h(r\cos\theta)\cos\theta\, d\theta}{[\sin^2\theta + (m+1)^2\cos^2\theta]^2} \\ &\geqslant \frac{1}{\sqrt{2c}} \int_{-\pi/4}^{0} \frac{h(r\cos\theta)\cos\theta\, d\theta}{[\sin^2\theta + (m+1)^2\cos^2\theta]^2}. \end{aligned}$$

For $\theta \in [-\pi/4, 0]$, and $c$ large enough, by (4.3)

$$r\cos\theta \geqslant \frac{\sqrt{2}}{2}\,\frac{\sqrt{2c}}{m+1} = \frac{\sqrt{c}}{m+1}, \qquad h(r\cos\theta) \geqslant \frac{\pi}{4}.$$

It is clear now that

$$\tau(c) - \frac{2\pi}{m+1} \geqslant \frac{1}{\sqrt{2c}}\,\frac{\pi}{4}\,\frac{\sqrt{2}}{2}\,\frac{\pi/4}{(m+1)^4} = \frac{\pi^2}{32(m+1)^4\sqrt{c}}. \tag{4.5}$$

On the other hand,

$$0 \leqslant \frac{1}{r} h(r \cos \theta) \cos \theta \leqslant \frac{\pi}{2} \frac{1}{r} \leqslant \frac{(m+1)\,\pi}{2\sqrt{2c}}.$$

For $c$ large enough,

$$\sin^2 \theta + (m+1)^2 \cos^2\theta - \frac{1}{r} h(r \cos \theta) \cos \theta \geqslant \frac{1}{2}.$$

From (4.3) and (4.4), we obtain

$$\tau(c) - \frac{2\pi}{m+1} \leqslant \int_{-2\pi}^{0} \frac{(m+1)\,\pi\, d\theta}{2\sqrt{2c}\,\frac{1}{2}(\sin^2 \theta + (m+1)^2 \cos^2 \theta)} \leqslant (m+1)\sqrt{\frac{2}{c}}\,\pi^2. \tag{4.6}$$

The inequalities (4.5), (4.6) show that

$$\tau(c) = \frac{2\pi}{m+1} + O\left(\frac{1}{\sqrt{c}}\right).$$

*Remark.* The condition

$H_4$.

$$\tau(c) \neq \frac{2\pi}{m} + O\left(\frac{1}{\sqrt{c}}\right) \text{ for any positive integer } m \text{ as } c \to \infty$$

seems unconvenient to check; however, by Lemma 2.4

$$\liminf_{c \to \infty} \tau(c) \geqslant \frac{4}{\sqrt{A_0}} \arcsin \sqrt{\frac{A_0}{K}} = \tau_0.$$

Thus, we only need to check those $m$ for which $2\pi/m \geqslant \tau_0$.

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